Question 1

A study of traffic using a tunnel showed that the following speed-concentration relationship applies:

$$u = 17.2 \ln\left(\frac{228}{k}\right) mi/h \tag{1.1}$$

a) Derive and plot the relation between the flow q and the concentration k
b) Find the capacity of the tunnel
c) Calculate the speed and concentration at capacity
d) Find the jam concentration

Solution

a) From Eq. (1.1), by substitution into $q = uk$, one obtains

$$q = uk = 17.2k \ln\left(\frac{228}{k}\right) = 17.2k\left(\ln 228 - \ln k\right) \tag{1.2}$$

Given Eq. (1.2), the relationship between the concentration k and the flow q is depicted in Figure 1.1.

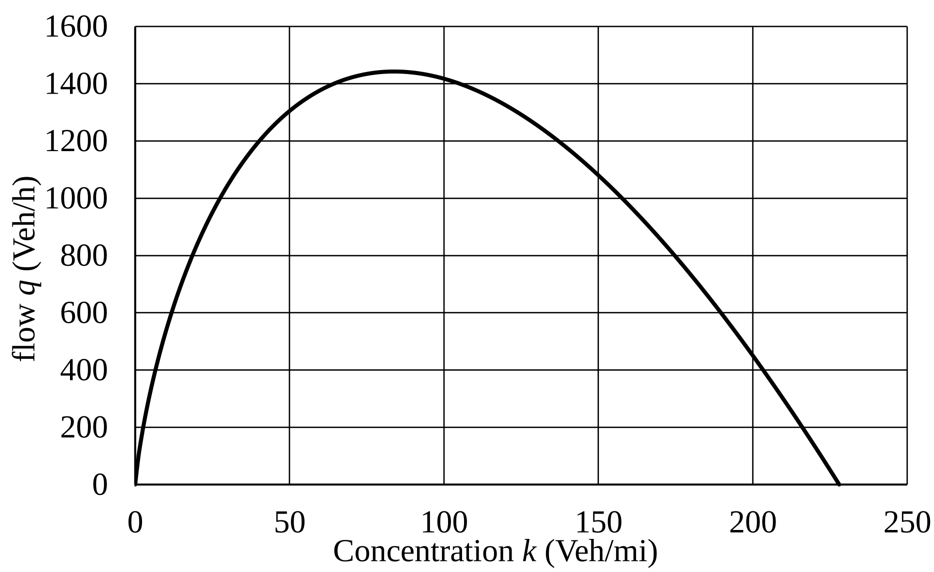

Figure 1.1 Relationship between the concentration and the flow

b) Recalling the capacity of the tunnel occurs when the slope of curve $q-k$ equals to zero. Take derivative in the Eq. (1.2), one obtains,

$$\frac{dq}{dk}=17.2\left[k\left(-\frac{1}{k}\right)+\left(\ln 228-\ln k\right)\right]=0 \tag{1.3}$$

Since $k\neq 0$, Eq. (1.3) can be rearranged as follows,

$$\ln k_{\max}=\ln 228-1=4.43 \tag{1.4}$$

Thus, the capacity $k_{\max}$ can be solved through the inverse function of the natural logarithm as follows

$$k_{\max}=e^{4.43}\approx 84\,\text{veh/mi} \tag{1.5}$$

From $k_{\max}=84$, by substitution into Eq. (1.2), one obtains,

$$q_{\max}=uk_{\max}=17.2k_{\max}\ln\left(\frac{228}{k_{\max}}\right)=17.2(84)\left(\ln\frac{228}{84}\right)=1445\,\text{veh/h} \tag{1.6}$$

c) The maximum concentration was provided in the previous solution for Question 1-b as $k_{\max}=84\,\text{veh/mi}$. By rearranging the equation $q=uk$, one obtains,

$$u_{\max}=\frac{q_{\max}}{k_{\max}}=\frac{1445}{84}=17.2\,\text{mi/h} \tag{1.7}$$

d) Recalling the jam concentration occurs when the speed equals to zero (i.e. $u=0$), by substituting $u=0$ into Eq. (1.1), one obtains,

$$17.2\ln\left(\frac{228}{k}\right)=0 \tag{1.8}$$

Since $\ln 1=0$, the jam concentration k_{jam} is calculated as $k_{jam}=228\,\text{veh/mi}$.

Question 2

The following data were taken on a highway:

Table 1.1 Date for the speed and corresponding concentration

u	52	34	36	22	21	mi/h
k	8	41	48	70	105	veh/mi

a) Estimate the free-flow speed assuming $q = AkB^k$

b) Calculate the capacity of the highway

c) Find u_m and k_m

Solution

a) Dividing the equation $q = AkB^k$ by the concentration k, the speed u can be obtained as follows

$$u = \frac{q}{k} = AB^k \tag{1.9}$$

Take the natural logarithm on both side of Eq. (1.9), one obtains

$$\ln u = \ln\left(AB^k\right) = \ln A + \ln\left(B^k\right) = \ln A + k\ln B \tag{1.10}$$

Equations (1.10) suggests a linear relationship between $\ln u$ and k since $\ln A$ and $\ln B$ are constants. Thus, a linear form for the relationship between $\ln u$ and k is adopted as follows by replacing $\ln u$ by Y

$$Y = C_1 + C_2 k \tag{1.11}$$

in which, $Y = \ln u$, C_1 and C_2 are constants and given by $\ln A$ and $\ln\ln B$, respectively.

By adopting a linear regression on the data in Table 1.1 (Fig 1.2), the Eq. (1.11) can be expressed as follows

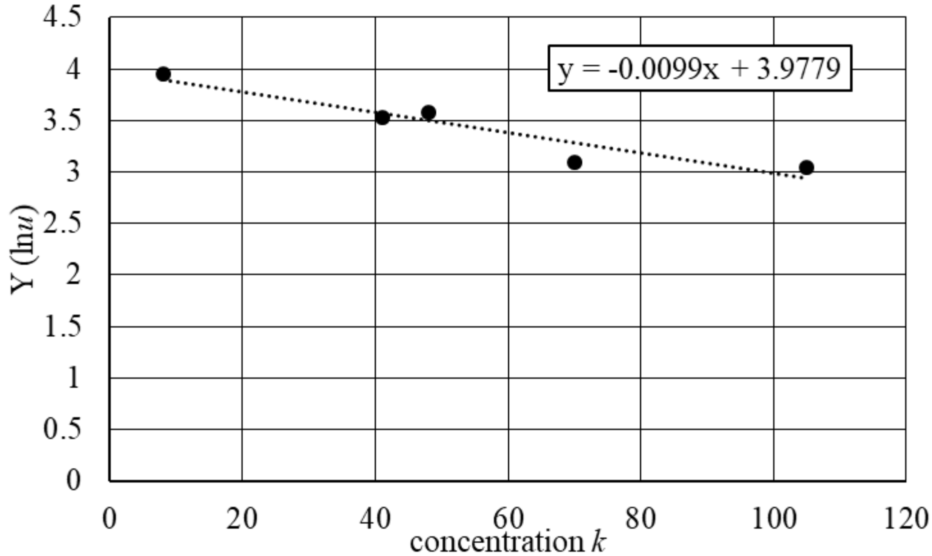

Figure 1.2 Linear regression result for $\ln u$ and the concentration k

$$Y = C_1 + C_2 k = 3.98 - 0.0099k \tag{1.12}$$

Thus, one obtains

$$\begin{aligned} \ln A &= 3.98 \\ \ln B &= -0.0099 \end{aligned} \tag{1.13}$$

Solve Eq. (1.13) for the constants A and B, one obtains

$$\begin{aligned} A &= 53.5 \\ B &= 0.99 \end{aligned} \tag{1.14}$$

Thus, the speed u and flow q can be expressed in terms of A and B as follows

$$\begin{aligned} u &= 53.5(0.99)^k \\ q &= uk = 53.5k(0.99)^k \end{aligned} \tag{1.15}$$

Given $u = 53.5(0.99)^k$, the free-flow speed can be calculated when $k \to 0$ as follows

$$u_f = 53.5 \times (0.99)^0 = 53.5\,\text{mi/h} \tag{1.16}$$

b) By taking derivative on $q = 53.5k(0.99)^k$, one obtains

$$\frac{dq}{dk} = 53.5(0.99)^k + 53.5k(0.99)^k \ln(0.99) \tag{1.17}$$

To find the value for k to equating Eq. (1.17) to zero (i.e. $dq/dk = 0$), the relationship between dq/dk and k is presented in Figure 1.3. It is found that the intersection of the $dq/dk - k$ curve with the k -axis is approximately 100. Thus, the maximum concentration k_m is found as $k_m = 100\,\text{veh/mi}$. The corresponding $q_{\max}$ is calculated by substituting $k_m = 100\,\text{veh/mi}$ in Eq. (1.15), one obtains

$$q_m = 53.5k_m(0.99)^{k_m} = 53.5(100)(0.99)^{100} = 1958\,\text{veh/h} \tag{1.18}$$

Thus, the corresponding u_m is computed as follows

$$u_m = \frac{q_m}{k_m} = \frac{1958}{100} = 19.58\,\text{mi/h} \tag{1.19}$$

(The procedure for solving k_m in Eq. (1.17) varies. You can do iteration trial for value of k in Excel to find the closest value for k_m when $dq/dk \to 0$. Or you can just use MATLAB to solve the equation directly.)

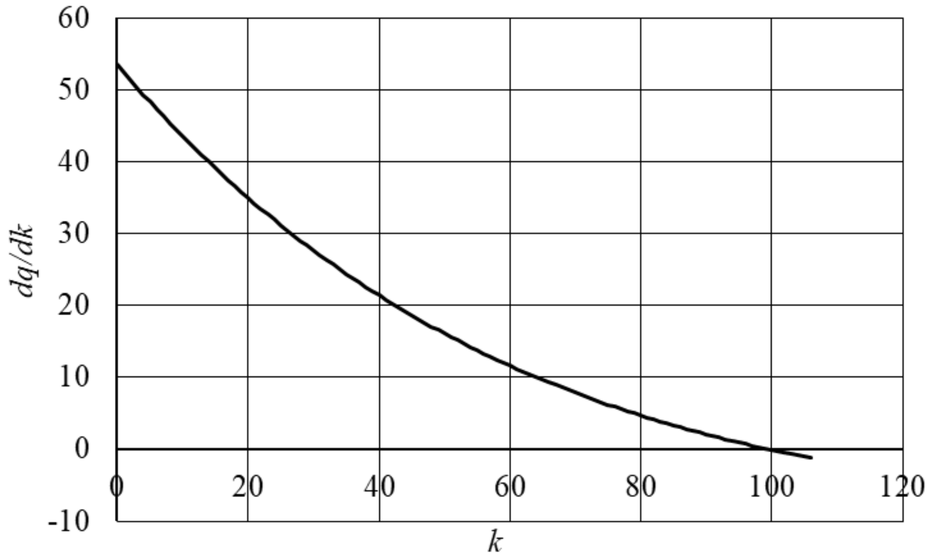

Figure 1.3 Relationship between dq/dk **and** k